Copying of quantum information by means of a quantum amplifier

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ABSTRACT

Information aspects of copying quantum states via stimulated emission of an optical quantum amplifier are considered. It is shown that the measurable information very rapidly decreases after amplification of a single photon up to a level of several photons. Spontaneous emission, which leads to such behavior, is also discussed.

Keywords: quantum amplification, quantum information, no-cloning principle

1. INTRODUCTION

This paper was initiated thanks to the various discussions of the no-cloning principle within the laser physics community. Despite the fact that this principle, which reads as *a priory* unknown quantum state cannot be cloned independently of its nature [1], is well understood within the quantum information community and can be formally proofed just in two lines, many people from the field of laser physics and quantum optics still wonder whether a quantum amplifier can break this principle or not. To be more specific, let us consider polarization of a photon as an information carrier and, therefore, the parameter that can be cloned. It is widely believed that a quantum amplifier produces due to the stimulated emission *the same* photons as the initial one. From this point of view if one sends a photon at the input of a quantum amplifier, in the output we will have several exact copies of the incident photon and this obviously violates the no-cloning principle.

This paradox can be easily solved if one takes into account unavoidable presence of spontaneous emission during the process of amplification of a photon, which readily leads to imperfections in the "cloning" process. Along with the generated photons with correct polarization, i.e. polarization of the initial photon, there always exist additional photons with incorrect polarization. Fidelity of such non-perfect cloning, defined as a relative number of output photons with correct polarization, must always be lower than unit. However, under certain assumption about the quantum amplifier—for instance, assumption about symmetry under unitary transformations of the initial photon (or system of several photons)—fidelity of the amplified signal [2, 3] is equal to the fidelity of optimal cloning [4–6].

For further analysis of information properties of a quantum amplifier one needs to define its model. Without any loss of generality, we will model quantum amplifier as an ensemble of Λ-systems with two degenerate ground states $|g_1\rangle$ and $|g_2\rangle$ and an excited state $|e\rangle$. Transitions between ground states and the excited one are connected with two electromagnetic polarization modes a_1 and a_2, which define the Hilbert space of the initial photon.

Hamiltonian of interaction between initial photon (or, generally speaking, electromagnetic field) with the quantum amplifier formed of K Λ–systems has the following form:

$$\mathcal{H} = \gamma \left(a_1^+ \sum_{k=1}^{K} |g_1^k\rangle \langle e^k| + a_2^+ \sum_{k=1}^{K} |g_2^k\rangle \langle e^k| \right) + \text{h.c.}, \tag{1}$$

where γ is the coupling constant of a Λ–system with the electromagnetic field. The physical meaning of this interaction Hamiltonian is rather clear: a new photon with the given polarization a_i^+ is created if a Λ-system switches from the excited state $|e\rangle$ on to the ground state $|g_i\rangle$.

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The general expression for the fidelity $F(N \to M)$ of amplification of N photons to $M > N$ was derived in Ref. [2,3] and reads as

$$F(N \to M) = \frac{MN + M + N}{M(N+2)}. \tag{2}$$

In the limiting case of amplification of a single photon ($N = 1$) to infinite number of output photons $N = \infty$, i.e., up to a classical signal, this expression transforms to $F(1 \to \infty) = 2/3$. The fact that $F(1 \to \infty)$ is higher than $1/2$ means that the majority of photons generated by the quantum amplifier have statistically correct polarization, which leaves a faint hope that extracting information about polarization of the initial photon would be possible. In fact, it is still unclear what does the value of fidelity more than $1/2$ mean in sense of extracting information from it. In this paper, we will clarify this and other questions related to the information, which one can actually retrieve from a single photon with the help of a quantum amplifier.

The paper is organized as follows. We start in Sec. 2 with consideration of the no-cloning principle and related limitation for the possible copying of information. Then, in Sec. 3 we consider measurable information after quantum amplification in detail and, finally, conclusions are given in Sec. 4.

2. NO-CLONING AND NO-COPYING PRINCIPLES

The formal proof of the no-cloning principle can be outlined as follows. Let us consider cloning of two arbitrary states $|\phi\rangle$ and $|\psi\rangle$. The result of the cloning can be represented as a unitary transformation that produces clones of these states from an initial blank state $|0\rangle$:

$$\begin{aligned} |\phi\rangle\,|0\rangle &\to |\phi\rangle\,|\phi\rangle, \\ |\psi\rangle\,|0\rangle &\to |\psi\rangle\,|\psi\rangle\,. \end{aligned} \tag{3}$$

Due to the unitarity of the cloning transformation the inner product of the initial joint system $A + B$ must be preserved:

$$\langle\phi\,|\psi\,\rangle = \langle\phi\,|\psi\,\rangle\,\langle\phi\,|\psi\,\rangle, \tag{4}$$

which can be fulfilled in only two cases: $\langle\phi\,|\psi\,\rangle = 0$ and $\langle\phi\,|\psi\,\rangle = 1$, which means cloning of the state from an orthogonal set. As a result, we can see that the perfect cloning is impossible if the initial state belongs to the nonorthogonal set of states belonging to the same Hilbert space.

This statement can be enforced by the no-copying principle [7], which tells that the perfect clone cannot be produced and even does not exist. This means that for two separate systems A and B the results of an arbitrary measurement of one of the systems cannot be always equal to the result of the same measurement made with another system.

To prove this statement, let us consider the divergency operator

$$\hat{C}_{AB} = \int \left(|\alpha\rangle_A \, \langle\alpha|_A \otimes \hat{1}_B - \hat{1}_A \otimes |\alpha\rangle_B \, \langle\alpha|_B\right)^2 d^2V_\alpha, \tag{5}$$

where dV_α is the volume differential on the Bloch sphere for the Hilbert space of the bipartite system A and B. It has been shown that this divergency operator equals to the following one [7]:

$$\hat{C}_{AB} = 2\left[\frac{1}{3}\left(||\Psi^+\rangle\!\rangle\langle\!\langle\Psi^+|| + ||\Phi^-\rangle\!\rangle\langle\!\langle\Phi^-|| + ||\Phi^+\rangle\!\rangle\langle\!\langle\Phi^+||\right) + ||\Psi^-\rangle\!\rangle\langle\!\langle\Psi^-||\right], \tag{6}$$

where $||\Psi\rangle\!\rangle^\pm$ and $||\Phi\rangle\!\rangle^\pm$ are the Bell states. Therefore, the average divergence of all indicative projectors, i.e., quantum variables with the possible values 0 and 1 of the bipartite system A and B, has ever nonzero value exceeding $2/3$.

It is worth to note here that despite the fact that the perfect copying does not exist, the perfect anticopying exists nevertheless. To proof this, let us consider the anticopying operator

$$\hat{A}_{AB} = \int (|\alpha\rangle_A \langle\alpha|_A \otimes \hat{1}_B - \hat{1}_A \otimes |\tilde{\alpha}\rangle_B \langle\tilde{\alpha}|_B)^2 d^2V_\alpha, \tag{7}$$

which is similar to the copying operator (5), but the opposite states $\tilde{\alpha}$ of the B-system are compared with the states α of the A-system. Then, one can easily find that if the bipartite state is antisymmetric,

$$\hat{A}_{AB} |\Psi\rangle^- = 0, \tag{8}$$

which means that the antisymmetric state provides perfect anti-correlations of the subsystems. This is a well-known fact in nuclear decay experiment, for instance: if the initial system has zero spin and resulting particles have spin of 1/2, then they are contrarily oriented.

3. MEASURABLE INFORMATION AFTER QUANTUM AMPLIFICATION

Keeping in mind equivalence of perfect copying to an equality of mutual information to unit, let us now answer the question of how much information can be extracted, or measured, with the help of a quantum amplifier.

The most natural quantitative measure for measurable information is the classical Shannon information functional

$$I_{AB} \equiv S[P(x)] - S[P(x|y)], \tag{9}$$

where $S[P(x)]$ is the standard Shannon entropy $S[P(x)] = -\sum P(x) \log_2 P(x)$, $P(x)$ and $P(x|y)$ are the unconditional (*a priori*) and conditional (*a posteriori*) probability distributions. Due to the fact that *a priori* entropy depends only on a set of possible states of the initial photon and can be treated as fixed, the measurable information determines only by the conditional entropy, or a given method of the measuring procedure.

For simplicity, we will consider a case of amplification of one initial photon to M output photons when a polarization of the initial photon belongs to a fixed set of polarizations, which can be taken, for example, the same as for the BB84 quantum key distribution protocol (vertical, horizontal, and $\pm 45°$ linear polarizations). Quantum amplification in this case can be considered as a possible strategy of eavesdropping. For this case expression (2) transforms to

$$F(1 \to M) = \frac{2}{3} + \frac{1}{3M}. \tag{10}$$

Experimental setup for retrieving classical information is similar to that one used for the intercept-resend strategy of eavesdropping and is shown in Fig. 1.

This experimental setup acts as follows. For measuring only vertical/horizontal polarization part I of this setup is to be used only. If one sends single photons, one by one without amplification, the photodetector 1 will always click and the photodetector 2 will never click (of course, under assumption of their perfect sensitivity and zero dark count rate). Thus, we can perfectly measure the polarization of a single photon, i.e., the measurable information reaches its maximal value, which for the given orthogonal set of states is equal to 1 bit.

However, if polarization of the initial photon belongs to a nonorthogonal set, and can be with equal probability either verticlal/horizontal or $\pm 45°$ polarization, we should use both parts of the experimental setup. With equal probabilities, the initial photon will pass through the beamsplitter to the part I or will be reflected to the part II, and accessible information I_{access} in this case is equal to 0.5 bit because in one half of the cases we will perfectly know polarizations and in another half of the cases we will have no information about the polarization at all.

Let us now analyze the results of using a quantum amplifier. After amplification of a single photon to M output photons the probability of obtaining each photon in the same polarization mode as that of initial photon is equal to $F(1 \to M)$ (see Eq. (10)), and in the orthogonal polarization mode—$Q(1 \to M) = 1 - F(1 \to M)$. The probability of having each output photon in the additional unbiased modes is equal to 1/2.

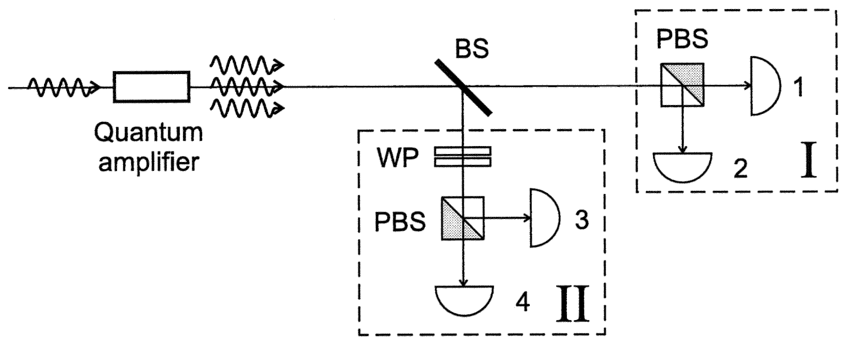

Figure 1. Experimental setup for retrieving information from a single photon. BS are the beamsplitters, PBS are the polarizing beamsplitters, WP is a waveplate to switch between vertical/horizontal and ±45° polarizations, and 1–4 are the photodetectors. Part I of the experimental setup measures vertical/horizontal polarizations, whereas part II — ±45° polarizations.

Therefore, we have *a posteriori* probability distribution $P(x|y)$ of obtaining output photon in x polarization mode, whereas the initial photon was in y mode:

$$P(x|y) = \begin{pmatrix} F(1 \to M) & Q(1 \to M) & 1/2 & 1/2 \\ Q(1 \to M) & F(1 \to M) & 1/2 & 1/2 \\ 1/2 & 1/2 & F(1 \to M) & Q(1 \to M) \\ 1/2 & 1/2 & Q(1 \to M) & F(1 \to M) \end{pmatrix}, \tag{11}$$

where $x, y = \overline{1,4}$ denotes consequently vertical, horizontal, +45°, and −45° polarization modes.

The corresponding measurable information (9) is shown in Fig. 2, which clearly indicates that the measurable information I very quickly decreases from its maximal value of 0.5 bit, corresponding to the case of no amplification, to its lowest value of $5/6 - \ln 3/\ln 4 \simeq 0.04$ bit, corresponding to the case of infinite number of output photons, i.e., entirely classical signal. This means that the quantum amplification results in dramatic decrease of measurable information and the best way to retrieve information from a single photon is to measure it without amplification.

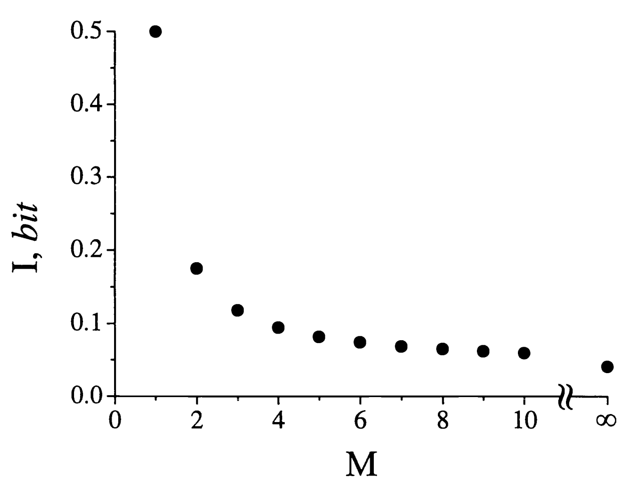

Figure 2. The measurable information I versus the number of output photons M after amplification of a single photon.

This result can be easily understood with the presence of spontaneous emission during the amplification, which acts as a source of additional noise. Taking it into account does not require any additional formalism, because it is just an inalienable part of the stimulated emission. This leads to the fact that along with the "right" photons (i.e., with the same polarization as that one of the initial photon) produced in the process of stimulated emission* there always be produced some "wrong" photons due to the spontaneous emission. In other words, spontaneous emission here reflects the fact that the amplification fidelity is not equal to the unit, or describe the immanent quantum fluctuations present in the process of amplification.

It is worth to note here that in the simplest case of generation of one photon by stimulated emission by only one atom, this atom cannot interact directly with the *exact* state $|\psi\rangle$ of the photon, but with the state that already includes its internal quantum uncertainty. This means that if the input photon has another state $|\phi\rangle$, the result of interaction of the atom with this photon will be the same as with the previous photon with the probability $|\langle\psi|\phi\rangle|^2$.

As a simple model, which increases the noise during amplification, let us consider a consecutive chain of Λ-systems, each of them produces only one additional photon stimulated by the randomly chosen photon at their input. As a result, we will have two photons after the first Λ-system, three after the second one, and so on. Each Λ-system in the sequence will generate due to the quantum uncertainty of the sates of the photons an additional quantum noise. This leads in additional errors with respect to the initial photon. These errors will be amplified along the chain with the signal amplification and, as result of such amplification, we will have essential part of "wrong" photons at the output of the amplifier.

Finally, let us clarify a question if the quantum amplification can be considered as a possible eavesdropping strategy. The answer is yes, it can be used as a possible eavesdropping strategy, but as a quite poor one: even far-from-the-optimal intercept–resend strategy extracts accessible information, which is the upper bound for the amount of the measurable information after amplification.

4. CONCLUSIONS

In conclusion, we studied the information aspects of the light amplification with its possible use as an eavesdropping strategy in quantum cryptography. We clarified the information picture of the amplification process and derived its complete information characteristics. It is shown that the measurable information very rapidly decreases after amplification of a single photon up to a level of several photons due to the inevitable presence of spontaneous emission, which leads to such behavior.

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*Note that in the process of stimulated emission the generated photons are always the "right" photons, i.e., the photons with the same polarization as that one of the initial photon.